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Core material from this paper was also subsequently published with new exposition in the journal article "Naturman, Colin. Countability Properties in Interior Algebras. Quaestiones Mathematicae 17.2 (1994): 189-204."

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Abstract: The intervals in an interior algebra **A** can be turned into interior algebras called interval algebras. Generalizations of homomomorphisms, called topomorphisms, are introduced and the interval algebras of **A** are shown to be precisely the principal quotients of **A** in the category of interior algebras and topomorphisms. The interval algebras of **A** obtained from principal ideals generated by open elements are shown to be precisely the principal homomorphic images of **A**.

AMS Classification: Primary 06E25, Secondary 06D99

Introduction

In this paper we introduce the concept of interval algebras of interior algebras (Definition 2.2) as a generalization of the concepts of ideal algebras of interior algebras [4] and of interval algebras of interior algebras generated by open elements [1]. We also introduce the concept of a topomorphism (Definition 1.1) which is a Boolean algebra homomorphism between interior algebras that preserves open elements. The main result of this paper is the characterization of interval algebras as being (up to isomorphism) the principal quotients of interior algebras in the category of interior algebras and topomorphisms.

In §0 we summarize the preliminaries needed for this paper. In §1 we introduce the maps between interior algebras known as topomorphisms (Definition 1.1). The definition of a topomorphism is such that the category of interior algebras and topomorphisms, **Int***, is isomorphic to the category of generalized topological spaces and their homomorphisms. We briefly investigate embeddings and quotient maps in **Int*** (Theorem 1.3 – Proposition 1.6). In §2 we define interval algebras (Proposition 2.1, Definition 2.2), and present some basic results. Of particular importance is Theorem 2.7 which shows that an interval algebra of an interior algebra is a quotient of that interior algebra in the category **Int***. §3 is devoted to showing that up to isomorphism, the interval algebras of an interior algebra **A** are precisely the principal quotients of **A** in **Int***. Theorem 3.1 shows that every principal quotient of a generalized topological space is itself a generalized topological space whose corresponding interior algebra **A** is isomorphic to an interval algebra and moreover, every interval algebra in **A** is isomorphic to an ideal algebra in **A** and a filter algebra in **A**. Corollary 3.3 characterizes those interval algebras in **A** which are principal homomorphic images of **A**. Theorem 3.7 combines the previous results to show to provide the complete characterization of interval algebras, ideal algebras and filter algebras of an interior algebra **A**, as being precisely the principal quotients of **A** in **Int***. Corollary 3.9 characterizes those principal quotients of **A** isomorphic to an ideal algebra generated by an open element (the open quotients), as being precisely the principal homomorphic images of **A**.

Note on Notation

Bold capitals will be used to denote structures. If **A**, **B**, **C**, ... are structures, the corresponding normal capitals A, B, C, ... will denote the underlying sets of these structures. If **A** is a structure, Con(**A**) will denote the congruence lattice of **A**. If $f : \mathbf{A} \to \mathbf{B}$ is a map between structures then ker(f) denotes the kernel of f, that is, the equivalence relation on A given by $(a,b) \in \ker(f)$ if and only if $f(a) = f(b)$.

0. Preliminaries

For basic results concerning interior algebras we refer the reader to [1], [3], [4] and [6]. We repeat some of the more important results and definitions below.

By an *interior algebra* we mean an algebraic structure $\langle B, {}^{I}, {}^{C} \rangle$ where:

(i) $\mathbf{B} = \langle B, \cdot, +, {}', 0, 1 \rangle$ is a Boolean algebra

(ii) ^{I} and ^{C} are unary operations satisfying:

- (a) $1^{I} = 1$
- (b) $a^{I} \leq a$
- (c) $a^{II} = a^{I}$
- (d) $(ab)^{I} = a^{I}b^{I}$
- (e) $a^{C} = a'^{I}{}'$ for all $a,b \in B$

The operations ^{I} and ^{C} are called the *interior operator* and *closure operator* respectively. For $a \in B$, a^{I} and a^{C} are known as the *interior* and the *closure* of a respectively.

We can generalize the principle of duality for Boolean algebras to interior algebras as follows: Given a sentence φ in a formal language for interior algebras define the dual of φ to be the sentence φ' obtained from φ by interchanging $\cdot$ and $+$, 0 and 1, and ^{I} and ^{C}. Then φ

holds if and only if φ' holds.

An element a of an interior algebra is called *open* if and only if $a^I = a$. The dual notion to that of an open element is that of a *closed* element: a is called closed if and only if $a^C = a$. Notice that a is open if and only if a' is closed. Moreover, a^I is always open and a^C is always closed. Elements which are both open and closed are called *clopen*. Given an interior algebra A we denote the set of open elements of A by A^O and the set of closed elements by A by $A^\square$. $0,1 \in A^O$ and A^O is closed under finite meets and joins and so $\langle A^O, \cdot, +, 0, 1 \rangle$ is a distributive 0,1–lattice. Moreover, for all $a,b \in A^O$, $a \to b := (a' + b)^I$ is the relative pseudocomplement of a with respect to b in this distributive 0,1–lattice. Thus we have a Heyting algebra $A^O = \langle A^O, \cdot, +, \to, 0, 1 \rangle$. Dually we have a Brouwerian algebra (dual Heyting algebra) $A^\square = \langle A^\square, \cdot, +, *, 0, 1 \rangle$ where for all $a,b \in A^\square$, $a * b := (a'b)^C$ is the dual relative pseudocomplement of a with respect to b.

By a *generalized topological space* [1] we mean a structure $\langle B, G \rangle$ where:

(i) $B = \langle B, \cdot, +, ', 0, 1 \rangle$ is a Boolean algebra

(ii) G is a unary relation satisfying:

  (a) $0,1 \in G$

  (b) G is closed under arbitrary joins.

  (c) G is closed under finite meets.

  (d) For all $a \in B$, $\Sigma \{b \in G : b \leq a\}$ exists.

G is said to be a *generalized topology* [1] in the Boolean algebra B. (If B is complete then condition (d) above holds automatically.) The defining conditions for a generalized topology are in a form which emphasizes the connection with topology: If $\mathcal{T}$ is a topology on a set X then $\mathcal{T}$ is a generalized topology in the power set Boolean algebra on X. The conditions can be simplified as the following proposition shows.

Proposition 0.1 [1]: Let B be a Boolean algebra. A subset $G \subseteq B$ is a generalized topology in B if and only if:

(a) $1 \in G$

(b) G is closed under binary meets.

(c) For all $a \in B$, $\max \{b \in G : b \leq a\}$ exists. □

Generalized topological spaces provide an alternative description of interior algebras: Given an interior algebra A let A^u denote the underlying Boolean algebra of A and put Gt A = $\langle A^u, A^O \rangle$. Given a generalized topological space $\langle B, G \rangle$ define operations ^{I} and ^{C} on B by $a^I = \max \{b \in G : b \leq a\}$ and $a^C = a'^{I'}$ or equivalently $a^C = \min \{b' : b \in G \text{ and } a \leq b'\}$ for all $a \in B$. Put Alg $\langle B, G \rangle = \langle B, {}^I, {}^C \rangle$. The we have:

Theorem 0.2 [1]: Let A be an interior algebra and let T a generalized topological space. Then:

(i) Gt A is a generalized topological space.

(ii) Alg T is an interior algebra.

(iii) Gt Alg T = T

(iv) Alg Gt A = A □

Thus interior algebras and generalized topological spaces are essentially the same things. In order to specify an interior algebra it suffices to specify its underlying Boolean algebra and its open elements, in other words, its corresponding generalized topological space. A useful way of representing finite interior algebras (with not much more than four atoms) and portions of infinite interior algebras is to give the lattice diagram of the underlying Boolean algebra and to indicate the open elements by circles.

1. Topomorphisms

If $f : A \longrightarrow C$ is an interior algebra homomorphism then $f : \mathrm{Gt}\ A \longrightarrow \mathrm{Gt}\ C$ is a homomorphism of Boolean algebras with added unary relation, since if $a \in A^O$, $f(a)^I = f(a^I) = f(a)$ whence $f(a) \in C^O$. However, not every homomorphism between generalized topological spaces gives rise to an interior algebra homomorphism. To see this consider the interior algebras A and C given by Fig. 1:

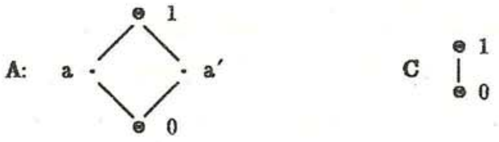

Fig. 1

Let $f : A \longrightarrow C$ be given by $f(0) = f(a') = 0$ and $f(a) = f(1) = 1$. Then $f : A \longrightarrow C$ is not an interior algebra homomorphism since $f(a)^I = 1^I = 1$ while $f(a^I) = f(0) = 0$. However, $f : \mathrm{Gt}\ A \longrightarrow \mathrm{Gt}\ C$ is easily seen to be a homomorphism of generalized topological spaces (considered as Boolean algebras with added unary relation). This leads us to make the following definition:

Definition 1.1: Let $f : A \longrightarrow C$ be a map between two interior algebras. f is called a *topomorphism* if and only if f is a Boolean algebra homomorphism with $f[A^O] \subseteq C^O$. □

Proposition 1.2: Let A and C be interior algebras. Then the following are equivalent:

(i) $f : A \longrightarrow C$ is a topomorphism.

(ii) $f : A \longrightarrow C$ is a Boolean algebra homomorphism with $f[A^{\square}] \subseteq C^{\square}$.

(iii) $f : \mathrm{Gt}\ A \longrightarrow \mathrm{Gt}\ C$ is a homomorphism. □

Let **Int** and **Int**$^+$ denote the categories of interior algebras and homomorphisms, and interior algebras and topomorphisms respectively. We see that up to isomorphism **Int**$^+$ is just the category of generalized topological spaces and their homomorphisms. We briefly investigate embeddings and quotient maps in **Int**$^+$. (See [2] pages 262 – 264 for basic results concerning embeddings. Quotient maps are the category theoretic duals of embeddings.)

Theorem 1.3: Let $f : A \longrightarrow C$ be an injective topomorphism. Then f is an embedding in **Int**$^+$ if and only if $f^{-1}[C^O] = A^O$.

Proof: Let f be an embedding in **Int**$^+$. Suppose there is an $a \in A$ with $f(a) \in C^O$ but $a \notin A^O$. Let S be the interior algebra given in Fig.2:

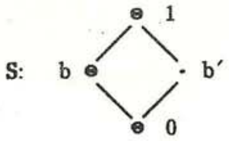

Fig. 2

Let $g : S \longrightarrow A$ be the Boolean algebra homomorphism with $g(b) = a$. Then fg is a topomorphism but g is not, a contradiction. Hence $f^{-1}[C^O] = A^O$. Now suppose $f^{-1}[C^O] = A^O$. Suppose D is an interior algebra and $h : D \longrightarrow A$ be a map with fh a topomorphism. f is a Boolean algebra embedding so h is a Boolean algebra homomorphism. Let $a \in D^O$. Then $fh(a) \in C^O$. Hence $h(a) \in A^O$ and so h is a topomorphism and the result follows. □

Corollary 1.4: Every embedding in **Int** is an embedding in **Int**$^+$.

Proof: Let $f : A \longrightarrow C$ be an **Int** embedding. If $a \in A$ with $f(a) \in C^O$, then $f(a^I) = f(a)^I = f(a)$ and so $a^I = a$, that is $a \in C^O$. Thus $f^{-1}[C^O] = A^O$. □

The converse of the above fails: Consider the interior algebras A and C given by Fig. 3:

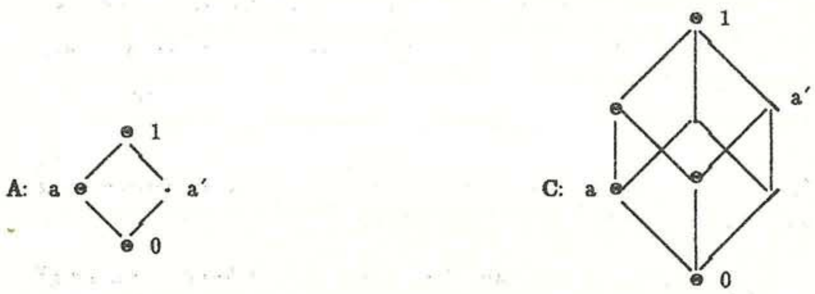

Fig. 3

The inclusion map of A in C is an embedding in **Int**$^+$ but it is not in **Int**.

Corollary 1.5: Isomorphisms in **Int**$^+$ and **Int** coincide. A topomorphism $f : A \longrightarrow C$ is an isomorphism iff f is bijective and $f[A^O] = C^O$. □

A nice characterization of quotient maps in **Int**$^+$ does not seem to be obtainable. However we have the following result:

Proposition 1.6: Let $f : A \longrightarrow C$ be a surjective topomorphism.

(i) If $f[A^O] = C^O$ then f is a quotient map in **Int**$^+$.

(ii) Every quotient map in **Int** (surjective homomorphism) satisfies (i) and is thus a quotient map in **Int**$^+$.

Proof: (i): Suppose $f[A^O] = C^O$. Let $g : C \longrightarrow D$ be a map such that gf is a topomorphism. Since f is a surjective Boolean algebra homomorphism, g is a Boolean algebra homomorphism. Let $b \in C^O$. Then there is an $a \in A^O$ with $b = f(a)$. Then $g(b) = gf(a) \in D^O$. Hence g is a topomorphism and the result follows. (ii): Let $f : A \longrightarrow C$ be a surjective homomorphism. Let $b \in C^O$. There is an $a \in A$ with $b = f(a)$. Then $a^I \in A^O$ with $f(a^I) = f(a)^I = b^I = b$. Thus $f[A^O] = C^O$. □

2. Interval Algebras

If A be an interior algebra and $a \leq b$ in A then [a,b] denotes the interval $\{c \in A : a \leq c \leq b\}$ in A. (a] and [a) denote the principal ideal [0,a] and the principal filter [a,0] respectively. We can define operations ${}^{\prime_{a,b}}$, ${}^{I_{a,b}}$ and ${}^{C_{a,b}}$ on [a,b] by $r^{\prime_{a,b}} = a + br'$, $r^{I_{a,b}} = a + b\cdot(b' + r)^I$ and $r^{C_{a,b}} = a + b\cdot(a'r)^C$. Let $\mathbf{[a,b]} = \langle [a,b], \cdot, +, {}^{\prime_{a,b}}, a, b, {}^{I_{a,b}}, {}^{C_{a,b}} \rangle$. We use **(a]** and **[a)** to denote **[0,a]** and **[a,1]** respectively and the operations in **(a]** and **[a)** will be denoted by ${}^{\prime_{,a}}$, ${}^{I_{,a}}$, ${}^{C_{,a}}$ and ${}^{\prime_{a,}}$, ${}^{I_{a,}}$, ${}^{C_{a,}}$ respectively. If we want to emphasize the interior algebra A in which **[a,b]** is taken we write A**[a,b]**.

Proposition 2.1: Let A be an interior algebra and let $a \leq b$ in A. Then **[a,b]** is an interior algebra.

Proof: We know that the structure $\langle [a,b], \cdot, +, {}^{\prime_{a,b}}, a, b \rangle$ is a Boolean algebra. We show that ${}^{I_{a,b}}$ is an interior operator with corresponding closure operator ${}^{C_{a,b}}$. (a): $1^{I_{a,b}} = a + b\cdot(b' + 1)^I = a + 1^I = a + 1 = 1$. (b): If $r \in [a,b]$ then $r^{I_{a,b}} = a + b\cdot(b' + r)^I \leq a + b\cdot(b' + r) = a + br = r$. (c): If $r \in [a,b]$ then $(r^{I_{a,b}})^{I_{a,b}} = a + b\cdot(b' + r^{I_{a,b}})^I = a + b\cdot(b' + a + b\cdot(b' + r)^I)^I = a + b\cdot(b' + a + (b' + r)^I)^I \geq a + b\cdot(b' + r)^{II} = a + b\cdot(b' + r)^I = r^{I_{a,b}}$. But from (b) we have $(r^{I_{a,b}})^{I_{a,b}} \leq r^{I_{a,b}}$ and so $(r^{I_{a,b}})^{I_{a,b}} = r^{I_{a,b}}$. (d): If $r,s \in [a,b]$ then $r^{I_{a,b}}\cdot s^{I_{a,b}} = (a + b\cdot(b' + r)^I)(a + b\cdot(b' + s)^I) = a + b\cdot(b' + r)^I\cdot b\cdot(b' + s)^I = a + b\cdot((b' + r)(b' + s))^I = a + (b' + rs)^I = (rs)^{I_{a,b}}$. (e): If $r \in [a,b]$ then $((r^{\prime_{a,b}})^{I_{a,b}})^{\prime_{a,b}} = a + b\cdot((r^{\prime_{a,b}})^{I_{a,b}})' = a + b\cdot(a + b\cdot(b' + r^{\prime_{a,b}})^I)' = a + ba'\cdot(b' + (b' + r^{\prime_{a,b}})^{I\prime}) = a + ba'\cdot(b' + r^{\prime_{a,b}})^{I\prime} = a + b\cdot(b' + r^{\prime_{a,b}})^{I\prime} = a + b\cdot(b' + a + br')^{I\prime} = a + b\cdot(b' + a + r')^{I\prime} = a + b\cdot(b' + a + r')^{\prime C} = a + b\cdot(ba'r)^C = a + b\cdot(a'r)^C = r^{C_{a,b}}$. □

Definition 2.2: If A is an interior algebra and $a \leq b$ in A then we call the interior algebra $[a,b]$ the *interval algebra* of A generated by a and b. The interior algebras $(a]$ and $[a)$ are called respectively, the *ideal algebra* and the *filter algebra* of A generated by a.

Ideal algebras generated by open elements were first introduced in [3] and were generalized to interval and filter algebras generated by open elements in [1]. Ideal algebras generated by arbitrary elements were introduced in [5].

Proposition 2.3: Let A be an interior algebra and let $a \leq c \leq d \leq b$ in A. Let $C = A[a,b]$. Then $C[c,d] = A[c,d]$.

Proof: Clearly the meet, join, top and bottom of $C[c,d]$ and $A[c,d]$ are the same and so it suffices to show that their complementation and interior operators are the same. For this consider $r \in [a,b]$. Then the complement of r in $C[a,b]$ is $c + dr'^{a,b} = c + d\cdot(a + br') = c + da + dbr' = c + a + dbr' = c + dr'$ which is the complement of r in $A[c,d]$. The interior of r in $C[c,d]$ is $c + d\cdot(d'^{a,b} + r)^{I_{a,b}} = c + d\cdot(a + b\cdot(d'^{a,b} + r)^I) = c + da + db\cdot(d'^{a,b} + r)^I) = c + a + d\cdot(d'^{a,b} + r)^I = c + d\cdot(a + bd' + r)^I = c + d\cdot(bd' + r)^I = c + d\cdot((b + r)(d' + r)^I = c + db\cdot(d' + r)^I = c + d\cdot(d' + r)^I$ which is the interior of r in $A[c,d]$. □

Proposition 2.4: Let $A = \Pi_I A_i$ be a product of interior algebras. Let $a = (a_i) \in A$ and let $b = (b_i) \in A$ with $a \leq b$. Then $\Pi_I[a_i,b_i] = [a,b]$. □

Under certain conditions ${}^{I_{a,b}}$ or ${}^{C_{a,b}}$ is 'independent' of either a or b:

Proposition 2.5: Let A be an interior algebra.

(i) $r^{I_{a,b}} = a + r^I$ for all $r \in [a,b]$ iff $b = a + b^I$.

(ii) $r^{I_{a,b}} = b\cdot(b' + r)^I$ for all $r \in [a,b]$ iff $a = b\cdot(b' + a)^I$.

(iii) $r^{C_{a,b}} = br^C$ for all $r \in [a,b]$ iff $a = ba^C$.

(iv) $r^{C_{a,b}} = a + (a'r)^C$ for all $r \in [a,b]$ iff $b = a + (a'b)^C$.

Proof: (i): Suppose $b = a + b^I$ and let $r \in [a,b]$. Then we have $r^{I_{a,b}} = a + b\cdot(b' + r)^I = a + (a + b^I)(b' + r)^I = a + a\cdot(b' + r)^I + b^I(b' + r)^I = a + (br)^I = a + r^I$. (ii): Suppose $a = b\cdot(b' + a)^I$ and let $r \in [a,b]$. Then $a \leq b\cdot(b' + r)^I$ and so $r^{I_{a,b}} = a + b\cdot(b' + r)^I = b\cdot(b' + r)^I$. Renaming a and b, (iii) and (iv) are the duals of (i) and (ii) respectively. □

Corollary 2.6: Let A be an interior algebra and let $a \leq b$ in A. Let $r \in [a,b]$.

(i) If b is open then $r^{I_{a,b}} = a + r^I$.

(ii) If a is open then $r^{I_{a,b}} = b\cdot(b' + r)^I$.

(iii) If a and b are then open $r^{I_{a,b}} = r^I$.

(iv) If b is closed then $r^{C_{a,b}} = a + (a'r)^C$.

(v) If a is closed then $r^{C_{a,b}} = br^C$.

(vi) If a and b are closed then $r^{C_{a,b}} = r^C$. □

Theorem 2.7: Let A be an interior algebra and let $a \leq b$ in A. Let $k : A \longrightarrow [a,b]$ be the surjective Boolean algebra homomorphism given by $k(r) = a + br$ for all $r \in A$. Then:

(i) $k[A^O] = [a,b]^O$ (equivalently $k[A^\square] = [a,b]^\square$)

(ii) k is a quotient map in $\mathbf{Int}^*$.

(iii) k is a homomorphism iff $a = ba^C$ and $b = a + b^I$.

Proof: (i): Let $r \in A^O$. Then $k(r)^{I_{a,b}} = (a + br)^{I_{a,b}} = a + b\cdot(b' + (a + br))^I = a + b\cdot(b' + a + r)^I \geq a + br^I = a + br = k(r)$. Thus $k(r)^{I_{a,b}} = k(r)$. Hence $k[A^O] \subseteq [a,b]^O$. Now consider $r \in [a,b]^O$. Then $r = r^{I_{a,b}} = a + b\cdot(b' + r)^I = k((b' + r)^I)$ and so $[a,b]^O \subseteq k[A^O]$ whence the result follows. (ii): This follows from (i) and Proposition 1.6 (i). (iii): Suppose k is a homomorphism. Then $a = a^{C_{a,b}} = k(a)^{C_{a,b}} = k(a^C) = a + ba^C = ba^C$ and $b = b^{I_{a,b}} = k(b)^{I_{a,b}} = k(b^I) = a + bb^I = a + b^I$. Conversely, suppose $a = ba^C$ and $b = a + b^I$. To show that k is a homomorphism we have to show that for any $r \in A$, $(a + br)^{I_{a,b}} = a + br^I$. Now

$(a + br)^{I_{a,b}} = (((a + br)^{\prime a,b})^{C_{a,b}})^{\prime a,b} = ((a + br')^{C_{a,b}})^{\prime a,b}$. Using Proposition 2.5 (iii) we then have $(a + br)^{I_{a,b}} = (b\cdot(a + br')^{C})^{\prime a,b} = (b\cdot(a^{C} + (br')^{C}))^{\prime a,b} = (ba^{C} + b\cdot(br')^{C})^{\prime a,b} = (a + b\cdot(br')^{C})^{\prime a,b} = a + b\cdot(br')^{C\prime} = a + b\cdot(br')^{\prime I} = a + b\cdot(b' + r)^{I} = a + (a + b^{I})(b' + r)^{I} = a + a\cdot(b' + r)^{I} + b^{I}(b' + r)^{I} = a + b^{I}(b' + r)^{I} = a + (br)^{I} = a + b^{I}r^{I} = (a + b^{I})(a + r^{I}) = b\cdot(a + r^{I}) = ba + br^{I} = a + br^{I}$, as required. □

Corollary 2.8: Let A be an interior algebra and let $a \leq b$ in A.

(i) If a and b are open then $[a,b]^{O} = A^{O} \cap [a,b]$.

(ii) If a and b are closed then $[a,b]^{\square} = A^{\square} \cap [a,b]$. □

Corollary 2.9: Let $k : A \longrightarrow [a,b]$ be as in Theorem 2.7. If a is closed and b is open then k is a homomorphism. □

Corollary 2.10: Let $f : A \longrightarrow C$ be a topomorphism and let $a \leq b$ in A. Let $g : [a,b] \longrightarrow [f(a),f(b)]$ be the restriction of f to [a,b]. Then g is a topomorphism and moreover:

(a) If f is injective, surjective or a homomorphism then so is g.

(b) If $f[A^{O}] = C^{O}$ then $g[[a,b]^{O}] = [f(a),f(b)]^{O}$. □

3. Characterization of Interval Algebras as Principal Quotients

Consider a generalized topological space $\mathbf{T} = \langle B, G \rangle$. Since $\mathbf{T}$ is a relational structure we may speak of congruences on $\mathbf{T}$. A congruence on $\mathbf{T}$ is just a Boolean algebra congruence on B. Thus if F is a filter in B we can form a congruence θ_F on $\mathbf{T}$, by putting $(a,b) \in \theta_F$ if and only if there is a $c \in F$ with $ac = bc$. Conversely, given a congruence θ on $\mathbf{T}$, putting $F_\theta = 1/\theta$ (the equivalence class of 1) gives a filter in B. Let $\mathbf{L}(B)$ denote the lattice of all filters in B and let $\mathbf{Con}(\mathbf{T})$ denote the lattice of all congruences on $\mathbf{T}$. Then the maps $F \longmapsto \theta_F$ and $\theta \longmapsto F_\theta$ are inverse isomorphisms between $\mathbf{L}(B)$ and $\mathbf{Con}(\mathbf{T})$. Let A be the interior algebra Alg $\mathbf{T}$ and let $\mathbf{Con}(A)$ denote the congruence lattice of A. $\mathbf{Con}(A)$ is a meet complete 0,1–sublattice of $\mathbf{Con}(\mathbf{T})$. If F is a filter in A (equivalently in B) then θ_F will be a congruence on A if and only if F is *open*, that is, $a^{I} \in F$ for all $a \in F$. The open filters of A form a meet complete 0,1–sublattice $\mathbf{L}^{O}(A)$ of $\mathbf{L}(B)$. The inverse isomorphisms $F \longmapsto \theta_F$ and $\theta \longmapsto F_\theta$ between $\mathbf{L}(B)$ and $\mathbf{Con}(\mathbf{T})$ restrict to inverse isomorphisms between $\mathbf{L}^{O}(A)$ and $\mathbf{Con}(A)$. The compact elements of $\mathbf{L}(B)$ are the principal filters $[a)$, $a \in B$, which form a 0,1–sublattice $\mathbf{L}_p(B)$ of $\mathbf{L}(B)$. The compact elements of $\mathbf{L}^{O}(B)$ are precisely the principal open filters $[a)$, $a \in G = A^{O}$, which form the 0,1–sublattice $\mathbf{L}_p^{O}(A) = \mathbf{L}_p(B) \cap \mathbf{L}^{O}(A)$ of $\mathbf{L}(B)$. We thus see that the compact congruences of $\mathbf{T}$ are precisely the principal congruences of $\mathbf{T}$, and they form a 0,1–sublattice $\mathbf{Con}_p(\mathbf{T})$ of $\mathbf{Con}(\mathbf{T})$. Also, the compact congruences of A are precisely the principal congruences of A and they form the 0,1–sublattice $\mathbf{Con}_p(A) = \mathbf{Con}_p(\mathbf{T}) \cap \mathbf{Con}(A)$ of $\mathbf{Con}(\mathbf{T})$. Moreover, the map $a \longmapsto \theta_{[a)}$ is a dual lattice isomorphism from B to $\mathbf{Con}_p(\mathbf{T})$ which restricts to a dual lattice isomorphism from A^{O} to $\mathbf{Con}_p(A)$. The map $a \longmapsto \theta_{[a')}$ gives an ordinary lattice isomorphism from B to $\mathbf{Con}_p(\mathbf{T})$ which restricts to a lattice isomorphism from $A^{\square}$ to $\mathbf{Con}_p(A)$.

Theorem 3.1: Let $\mathbf{T} = \langle B, G \rangle$ be an generalized topological space and let $A = \text{Alg}\ \mathbf{T}$. Let $a \leq b$ in A and let θ be the congruence $\theta_{[a'b)}$ on $\mathbf{T}$. Then the quotient structure $\mathbf{T}/\theta$ is a generalized topological space and moreover we have Alg $\mathbf{T}/\theta \cong [a,b] \cong (a'b] \cong [a + b')$.

Proof: The map $x \longmapsto xa'$ is a Boolean algebra isomorphism from $[a,b]$ to $(a'b]$. Suppose $x \in [a,b]^{O}$. Then by Theorem 2.7 (i) there is a $y \in A^{O}$ with $x = a + by$. Then $xa' = a'by$ and so by Theorem 2.7 (i), $xa' \in (a'b]^{O}$. Conversely, if $x \in (a'b]^{O}$ there is a $y \in A^{O}$ with $x = a'by$. Put $z = a + by$. Then $z \in [a,b]^{O}$ and $za' = x$. It follows that the map $x \longmapsto xa'$ is an interior algebra isomorphism from $[a,b]$ to $(a'b]$. Similarly, the map $x \longmapsto x + a + b'$ gives an isomorphism from $(a'b]$ to $[a + b')$. Thus $[a,b] \cong (a'b] \cong [a + b')$. We show that $\mathbf{T}/\theta$ is a generalized topological space with Alg $\mathbf{T}/\theta \cong (a'b]$. For this it suffices to show that $\mathbf{T}/\theta$ is isomorphic to Gt $(a'b]$ as Boolean algebras with added unary relation. We know that the map $x \longmapsto x/\theta$ gives a Boolean algebra isomorphism from Gt $(a'b]$ to $\mathbf{T}/\theta$.

Note that $T/\theta = \langle B/\theta, G/\theta \rangle$ where $G/\theta = \{a/\theta : a \in G\}$. Consider $x \in (a'b]^O$. Then there is a $y \in A^O = G$ with $x = ya'b$. Then $x/\theta = y/\theta$ since $xa'b = x = ya'b$ and $a'b \in [a'b)$. It follows that $x/\theta \in G/\theta$. Thus the map $x \longmapsto x/\theta$ is a homomorphism of Boolean algebras with added unary relation; we show that its inverse is also such a homomorphism. For this consider $X \in G/\theta$. Then there is a $y \in G$ with $X = y/\theta$. Put $x = ya'b$. Then $x \in (a'b]^O$ with $x/\theta = X$, whence the result follows.

Remark 3.2: Let $T = \langle B, G \rangle$ be an generalized topological space and let $A = \mathrm{Alg}\ T$. Let $a \le b$ in A and let θ be the congruence $\theta_{[a'b)}$ on T. Let $k : A \longrightarrow [a,b]$ be the quotient map of Theorem 2.7, given by $k(r) = a + br$. Then $\theta = \ker(k)$, the kernel of k.

Corollary 3.3: Let $T = \langle B, G \rangle$ be an generalized topological space and let $A = \mathrm{Alg}\ T$. Let $a \le b$ in A and let θ be the congruence $\theta_{[a'b)}$ on T.

(i) θ is an interior algebra congruence if and only if $a = ba^C$ and $b = a + b^I$.

(ii) If a is closed and b is open then θ is an interior algebra congruence.

Proof: (i): $\theta = \ker(k)$ where $k : A \longrightarrow [a,b]$ is the quotient map given by $k(r) = a + br$ for all $r \in A$, and so the result follows from Theorem 2.7 (ii). (ii): Clear from (i).

3.4 Corollary: Let A be an interior algebra and let $(a_i)_I$ and $(b_i)_I$ be families of elements of A. Suppose that for all $i \in I$, $a_i = b_i a_i^C$ and $b_i = a_i + b_i^I$. Let $f : A \longrightarrow \Pi_I[a_i,b_i]$ be given by $f(r) = (a_i + b_i r)_I$ for all $r \in A$. Then the following are equivalent:

(i) $f : A \longrightarrow \Pi_I[a_i,b_i]$ is a subdirect embedding of interior algebras.

(ii) $\Sigma_I a_i' b_i = 1$. □

Recall that a *partition of 1* in a Boolean algebra B is a subset $S \subseteq B$ which is pairwise disjoint (that is $a,b \in S$ and $a \neq b$ implies $ab = 0$) and has join $\Sigma S = 1$.

Corollary 3.5: Let A be an interior algebra. Let $n < \omega$ and let $a_1,\ldots,a_n,b_1,\ldots,b_n \in A$ with $a_i = b_i a_i^C$ and $b_i = a_i + b_i^I$ for $i = 1,\ldots,n$. Let $f : A \longrightarrow \Pi_I[a_i,b_i]$ be given by $f(r) = (a_1 + b_1 r ,\ldots, a_n + b_n r)$ for all $r \in A$. Then the following are equivalent:

(i) $f : A \longrightarrow \Pi_I[a_i,b_i]$ is an isomorphism.

(ii) $\{a_1'b_1,\ldots,a_n'b_n\}$ is a partition of 1. □

Definition 3.6: Let A be an interior algebra. An interior algebra C is called a *principal quotient* of A if and only if there is a principal congruence θ on $\mathrm{Gt}\ A$ such that $C \cong \mathrm{Alg}\ (\mathrm{Gt}\ A)/\theta$. □

Theorem 3.7: Let A and C be interior algebras. The following are equivalent:

(i) C is a principal quotient of A.

(ii) $C \cong (a]$ for some $a \in A$.

(iii) $C \cong [a)$ for some $a \in A$.

(iv) $C \cong [a,b]$ for some $a \le b$ in A.

Proof: By Theorem 3.1 (i) we immediately see that (ii), (iii), (iv) are equivalent and (iv) $\Longrightarrow$ (i). Now assume (i). There is a principal congruence $\theta = \mathrm{con}(c,d)$ on $\mathrm{Gt}\ A$ such that $C \cong \mathrm{Alg}\ (\mathrm{Gt}\ A)/\theta$. Put $a = cd + c'd'$. Then we see that $\theta = \mathrm{con}(a,1) = \theta_{[a)}$. Noting that $a = 0' \cdot a$ we see by Theorem 3.1 (i) that $C \cong (a]$. Thus (i) $\Longrightarrow$ (ii) and the result follows. □

Definition 3.8: Let C be a principal quotient of A. By Theorem 3.5 there is an $a \in A$ such that $C \cong (a]$. C is said to be an *open quotient* of A if and only if we can choose a to be an open element. Similarly we can define *closed quotients*, *clopen quotients* etc. □

Corollary 3.9: Let A and C be interior algebras. The following are equivalent:

(i) C is an open quotient of A.

(ii) C is a principal homomorphic image of A.

Proof: (i) $\Longrightarrow$ (ii) by Corollary 2.9 and Remark 3.2. (ii) $\Longrightarrow$ (i): Assume (ii). There is a principal interior algebra congruence θ on A such that $C \cong A/\theta$. Then $\theta = \theta_{[a)}$ for some $a \in A^O$. Noting that $a = 0' \cdot a$ and applying Theorem 3.1 we have $C \cong (a]$. □

Remark 3.10: Notice that direct factors of interior algebras are always clopen quotients: If $A = \Pi_I A_i$ is a product of interior algebras and $j \in I$ then $A_j \cong (a]$ where $a = (a_i)$ is given by $a_j = 1$ and $a_i = 0$ for $i \neq j$. □

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